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Abstract: In this paper we discuss second order optimality conditions in optimization problems subject to abstract constraints. Our analysis is based on various concepts of second order tangent sets and parametric duality. We introduce a condition, called second order regularity, under which there is no gap between the corresponding second order necessary and second order sufficient conditions. We show that the second order regularity always holds in the case of semi-definite programming.

Key-words: Second order optimality conditions, semi-definite programming, semi-infinite programming, tangent sets, Lagrange multipliers, cone constraints, duality

(Résumé : tsvp)

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Conditions nécessaires et suffisantes d'optimalité du deuxième ordre sous des contraintes abstraites

Résumé : Cet article présente des conditions d'optimalité du deuxième ordre pour des problèmes d'optimisation soumis à des contraintes abstraites. L'analyse est basée sur divers concepts d'ensemble tangents du deuxième ordre et sur la dualité paramétrique. Nous introduisons la condition de régularité du deuxième ordre sous laquelle il n'y a pas d'écart entre les conditions nécessaires et suffisantes du deuxième ordre. Nous montrons que cette condition est toujours satisfaite dans le cas de l'optimisation semi définie.

Mots-clé : Conditions d'optimalité du deuxième ordre, optimisation semi définie, optimisation semi-infinie, ensembles tangents, multiplicateurs de Lagrange, contraints de cône, dualité

AMS subject classifications 49K27, 90C30, 90C34.

1 Introduction

In this paper we investigate *necessary* and *sufficient* second order optimality conditions for an optimization problem in the form

$$(P) \qquad \operatorname*{Min}_{x \in X} f(x) \ \text{ subject to } \ G(x) \in K, \tag{1.1}$$

where X is a finite dimensional space, Y is a Banach space, K is a closed convex subset of Y, and the objective function $f : X \to I\!R$ as well as the constraint mapping $G : X \to Y$ are assumed to be twice continuously differentiable.

A number of optimization problems can be formulated in the form (1.1) in a natural way [7]. When Y is finite dimensional, say $Y = I\!R^p$, and $K = \{0\} \times I\!R_+^{p-q}$, the feasible set of (P) is defined by a finite number of equality and inequality constraints, and (P) becomes a nonlinear programming problem. As another example consider the space $Y = C(\Omega)$ of continuous functions $\psi : \Omega \to I\!R$, defined on a compact metric space Ω and equipped with the sup-norm $\|\psi\| := \sup_{\omega \in \Omega} |\psi(\omega)|$. Let $K = C_+(\Omega)$ be the cone of nonnegative functions, i.e.

$$C_+(\Omega) := \{\psi \in C(\Omega) : \psi(\omega) \geq 0, \ \forall\, \omega \in \Omega\}.$$

In that case the abstract constraint $G(x) \in K$ corresponds to $g(x, \omega) \geq 0$ for all $\omega \in \Omega$, where $g(x, \cdot) := G(x)(\cdot)$. If the set Ω is infinite, this leads to an infinite number of constraints and (P) becomes a semi-infinite programming problem. Yet another example is provided by semi-definite programming (see, e.g., [26] and references therein). There $Y = \mathcal{S}^n$ is the space of $n \times n$ symmetric matrices, and $K = \mathcal{S}_+^n$ is the cone of positive semi-definite matrices. Note that $\mathcal{S}_+^n$ can be represented in the form

$$\mathcal{S}_+^n = \{Z \in \mathcal{S}^n : \omega^t Z \omega \geq 0, \ \omega \in I\!R^n, \|\omega\| = 1\},$$

so that semi-definite programming can be considered in the framework of semi-infinite programming.

An alternative way of studying abstract optimality conditions is by considering optimization problems of the form

$$\operatorname*{Min}_{x \in X} F(H(x)), \tag{1.2}$$

where $F : Y \to \bar{I\!R}$ is a lower semicontinuous extended real valued convex function and $H : X \to Y$. This problem is equivalent to

$$\operatorname*{Min}_{(x,c) \in X \times I\!R} c \ \text{ subject to } \ (H(x), c) \in \operatorname{epi} F, \tag{1.3}$$

where $\operatorname{epi} F := \{(y, c) \in Y \times I\!R : F(y) \leq c\}$ is the epigraph of F, and hence it can be considered as a particular case of the problem (1.1). The converse is also true, that is, problem (1.1) can be represented in the form (1.2), so that both approaches essentially are equivalent.

Second order optimality conditions have been studied in numerous publications. In a sense, complete characterization of second order necessary and sufficient optimality conditions in the case of nonlinear programming (i.e. when the space Y is finite dimensional and the set K is polyhedral) were already given in Ioffe [11] and Ben-Tal and Zowe [1]. Second order necessary conditions in the general case using both approaches (1.1) and (1.2), and in the special case of semi-infinite programming, have been extensively studied (e.g. [3, 8, 10, 12, 14, 16]).

In this paper we investigate second order optimality conditions for an optimization problem given in the form (1.1), paying special attention to *sufficient* conditions. We introduce the notion of second order regularity for the constraint set K, under which the gap between necessary and sufficient conditions reduces to a change from weak to strict inequality, just as in the case of mathematical programming. We then exhibit a number of situations where second order regularity is satisfied. In particular we show that it is possible to give a complete characterization of second order conditions in the case of semi-definite programming.

For a function $h : Y \to I\!R$, we denote by $h'(y, d)$ its directional derivative

$$h'(y,d) = \lim_{t\downarrow 0} \frac{h(y+td) - h(y)}{t},$$

and by $h''_-(y; d, w)$ and $h''_+(y; d, w)$ its lower and upper second-order directional derivatives, respectively, i.e.

$$h''_-(y;d,w) = \liminf_{t\downarrow 0} \frac{h(y+td+\frac{1}{2}t^2 w) - h(y) - th'(y,d)}{\frac{1}{2}t^2},$$

$$h''_+(y;d,w) = \limsup_{t\downarrow 0} \frac{h(y+td+\frac{1}{2}t^2 w) - h(y) - th'(y,d)}{\frac{1}{2}t^2}.$$

Note that if the function $h(\cdot)$ is convex continuous, then (finite valued) directional derivatives $h'(y, d)$ exist for all $d \in Y$. We say that $h(\cdot)$ is second order directionally differentiable, at y in the direction d, if $h''_-(y; d, w)$ is equal to $h''_+(y; d, w)$ and is finite for all $w \in Y$. In this case the common value is denoted $h''(y; d, w)$.

By Y^* we denote the dual space of Y and by $\langle y^*, y\rangle$ the value $y^*(y)$ of the linear functional $y^* \in Y^*$ at $y \in Y$. For a set $T \subset Y$ we denote by $\sigma(\cdot, T)$ its support function, i.e. $\sigma(y^*, T) := \sup_{y\in T} \langle y^*, y\rangle$, and by $\mathrm{dist}(\cdot, T)$ the distance $\mathrm{dist}(y, T) := \inf_{z\in T} \|y - z\|$. By $Df(x)$ and $D^2 f(x)$ we denote the first and second order derivatives, respectively, of a function $f(x)$. We denote by $B_Y := \{y \in Y : \|y\| \le 1\}$ the unit ball in Y.

2 Tangent sets

In this section we discuss the notions of first and second order tangent sets on which our second order optimality conditions are based.

Let us first recall the notion of limits in the sense of Painlevé-Kuratowski for a multifunction $\Psi : X \to 2^Y$, from a normed space X into the set 2^Y of subsets of Y. The upper limit $\limsup_{x \to x_0} \Psi(x)$ is the set of points $y \in Y$ for which *there exists* a sequence $x_n \to x_0$ such that $y_n \to y$ for some $y_n \in \Psi(x_n)$. The lower limit $\liminf_{x \to x_0} \Psi(x)$ is the set of points $y \in Y$ such that for *every* sequence $x_n \to x_0$ it is possible to find $y_n \in \Psi(x_n)$ such that $y_n \to y$.

Let K be a closed convex subset of a Banach space Y. The (first-order) tangent set (cone) to K at a point $y \in K$ can be defined as follows

$$T_K(y) := \{h \in Y : \mathrm{dist}(y + th, K) = o(t),\ t \geq 0\}.$$

By the definition of lower set limits this can be written in the form

$$T_K(y) = \liminf_{t \downarrow 0} \frac{K - y}{t}. \tag{2.1}$$

It is well known that since K is convex, it is also true that

$$T_K(y) = \limsup_{t \downarrow 0} \frac{K - y}{t}. \tag{2.2}$$

Note that if K is a convex cone and $y \in K$, then $T_K(y) = \mathrm{cl}(K + [\![y]\!])$, where $[\![y]\!]$ denotes the linear space generated by vector y and "cl" stands for the topological closure in the norm topology of Y.

Similarly to (2.1) and (2.2) we consider second-order variations of the set K at a point $y \in K$ in a direction d. That is,

$$T_K^2(y, d) := \liminf_{t \downarrow 0} \frac{K - y - td}{\frac{1}{2}t^2}, \tag{2.3}$$

$$\mathcal{T}_K^2(y, d) := \limsup_{t \downarrow 0} \frac{K - y - td}{\frac{1}{2}t^2}. \tag{2.4}$$

We call $T_K^2(y,d)$ and $\mathcal{T}_K^2(y,d)$ the *inner* and *outer* second-oder tangent sets, respectively. Alternatively these tangent sets can be written in the form

$$T_K^2(y, d) = \left\{w \in Y : \mathrm{dist}(y + td + \tfrac{1}{2}t^2 w, K) = o(t^2),\ t \geq 0\right\},$$

$$\mathcal{T}_K^2(y, d) = \left\{w : \exists\, t_n \downarrow 0 \text{ such that } \mathrm{dist}(y + t_n d + \tfrac{1}{2}t_n^2 w, K) = o(t_n^2)\right\}.$$

It is clear from the above definitions that $T_K^2(y,d) \subset \mathcal{T}_K^2(y,d)$ and that these second-order tangent sets can be non empty only if $d \in T_K(y)$. Also both sets $T_K^2(y,d)$ and $\mathcal{T}_K^2(y,d)$ are closed and the set $T_K^2(y,d)$ is convex. The following example demonstrates that, unlike the first order tangent variations, the second order inner and outer tangent sets can be different.

Example 2.1 *Let us first construct a convex piecewise linear function $y = \eta(x)$, $x \in I\!R$, oscillating between two parabolas $y = x^2$ and $y = 2x^2$. That is, we construct $\eta(x)$ in such a way that $\eta(x) = \eta(-x)$, $\eta(0) = 0$ and for some monotonically decreasing to zero sequence x_k, the function $\eta(x)$ is linear on every interval $[x_{k+1}, x_k]$, $\eta(x_k) = x_k^2$ and the straight line passing through the points $(x_k, \eta(x_k))$ and $(x_{k+1}, \eta(x_{k+1}))$ is tangent to the curve $y = 2x^2$. It is quite clear how such a function can be constructed. For a given point $x_k > 0$ consider the straight line passing through the point (x_k, x_k^2) and tangent to the curve $y = 2x^2$. This straight line intersects the curve $y = x^2$ at a point x_{k+1}. Clearly $x_k > x_{k+1} > 0$. One can proceed the construction in an iterative way. It is easily proved that $x_k \to 0$. Define $K := \{(x, y) \in I\!R^2 : y \geq \eta(x)\}$. We have then that for the direction $d := (1, 0)$, $T_K^2(0, d) = \{(x, y) : y \geq 4\}$ and $\mathcal{T}_K^2(0, d) = \{(x, y) : y \geq 2\}$.*

We say that the set K is *second-order directionally differentiable*, at $y \in K$ in the direction d, if $T_K^2(y, d) = \mathcal{T}_K^2(y, d)$ (for various related concepts see [21]). This terminology is justified by the following result (cf. [8, Proposition 4.1]).

Proposition 2.1 *Suppose that the set K is defined in the form $K = \{y \in Y : h(y) \leq 0\}$ where $h(\cdot)$ is a continuous convex function. Let $h(y) = 0$, $h'(y, d) = 0$ and suppose that there exists $\bar{y}$ such that $h(\bar{y}) < 0$ (Slater condition). Then*

$$T_K^2(y, d) = \{w : h''_+(y; d, w) \leq 0\}, \tag{2.5}$$

$$\mathcal{T}_K^2(y, d) = \{w : h''_-(y; d, w) \leq 0\}. \tag{2.6}$$

Proof. We only show that (2.6) holds, since the proof of (2.5) is similar.

Let $w \in \mathcal{T}_K^2(y, d)$ and choose sequences $t_n \downarrow 0$ and $w_n \to w$ such that $y + t_n d + \frac{1}{2} t_n^2 w_n \in K$. Since $h(\cdot)$ is convex continuous, $h(\cdot)$ is locally Lipschitz continuous and hence

$$\tfrac{1}{2} t_n^2 h''_-(y; d, w) + o(t_n^2) = h(y + t_n d + \tfrac{1}{2} t_n^2 w_n) \leq 0.$$

It follows that $h''_-(y; d, w) \leq 0$.

Conversely, if $h''_-(y; d, w) < 0$, then for some $t_n \downarrow 0$,

$$h(y + t_n d + \tfrac{1}{2} t_n^2 w) = \tfrac{1}{2} t_n^2 h''_-(y; d, w) + o(t_n^2).$$

Consequently $h(y + t_n d + \frac{1}{2} t_n^2 w) < 0$ for n large enough, and hence $y + t_n d + \frac{1}{2} t_n^2 w \in K$, which implies that $w \in \mathcal{T}_K^2(y, d)$. Suppose now that $h''_-(y; d, w) = 0$. Given $\alpha > 0$, set $w_\alpha := w + \alpha(\bar{y} - y)$. Then by convexity of h we have

$$h(y + td + \tfrac{1}{2} t^2 w_\alpha) \leq (1 - \tfrac{1}{2} \alpha t^2) h(y + td + \tfrac{1}{2} t^2 w) + \tfrac{1}{2} \alpha t^2 h(\bar{y} + td + \tfrac{1}{2} t^2 w),$$

for $t \geq 0$ small enough such that $1 - \frac{1}{2} \alpha t^2 > 0$. Since $h(\bar{y}) < 0$ it then follows that for any $\alpha > 0$,

$$h''_-(y; d, w_\alpha) \leq h''_-(y; d, w) + \alpha h(\bar{y}) < 0 \tag{2.7}$$

and hence $w_\alpha \in \mathcal{T}_K^2(y, d)$. Since $\mathcal{T}_K^2(y, d)$ is closed, letting $\alpha \downarrow 0$ we obtain that $w \in \mathcal{T}_K^2(y, d)$ and the proof is complete. ∎

From this proposition it follows in particular that if the Slater hypothesis holds, then K is second-order directionally differentiable if and only if the level sets of $h''_-(y;d,\cdot)$ and $h''_+(y;d,\cdot)$ coincide for the 0 level. In particular, K is second-order directionally differentiable whenever $h(\cdot)$ is second-order directionally differentiable.

To close this section we state two results, which extend Proposition 3.1 and Theorem 3.1 in [8] to the case of outer second order tangent sets. We omit the proofs which are simple modifications of those in the cited reference.

Lemma 2.1 *For all $y \in K, d \in T_K(y)$ one has*

$$T_K^2(y,d) + T_{T_K(y)}(d) \subset T_K^2(y,d) \subset T_{T_K(y)}(d), \tag{2.8}$$

$$\mathcal{T}_K^2(y,d) + T_{T_K(y)}(d) \subset \mathcal{T}_K^2(y,d) \subset T_{T_K(y)}(d). \tag{2.9}$$

In particular, it follows that when $0 \in \mathcal{T}_K^2(y,d)$ then $\mathcal{T}_K^2(y,d) = T_{T_K(y)}(d)$, and when $0 \in T_K^2(y,d)$ all three sets coincide: $T_K^2(y,d) = \mathcal{T}_K^2(y,d) = T_{T_K(y)}(d)$.

The following formulas (2.11) and (2.12) provide a rule for computing the second order tangent approximations of the feasible set $G^{-1}(K)$ of (P), in terms of the second order tangent approximations of K. These formulas are valid under Robinson's constraint qualification [18],

$$0 \in \operatorname{int}\{G(x_0) + DG(x_0)X - K\}. \tag{2.10}$$

and can be proved by using the Robinson [19] - Ursescu [25] stability theorem (see [8]).

Proposition 2.2 *Let $x_0 \in G^{-1}(K)$ and suppose that Robinson's constraint qualification (2.10) holds. Then, for all $h \in X$,*

$$T_{G^{-1}(K)}^2(x_0,h) = DG(x_0)^{-1}\left[T_K^2(G(x_0), DG(x_0)h) - D^2G(x_0)(h,h)\right], \tag{2.11}$$

$$\mathcal{T}_{G^{-1}(K)}^2(x_0,h) = DG(x_0)^{-1}\left[\mathcal{T}_K^2(G(x_0), DG(x_0)h) - D^2G(x_0)(h,h)\right]. \tag{2.12}$$

3 Second order optimality conditions

In this section we derive second order necessary and sufficient optimality conditions for a problem (P) given in the form (1.1). With problem (P) are associated the Lagrangian

$$L(x,\lambda) := f(x) + \langle \lambda, G(x)\rangle, \quad \lambda \in Y^*,$$

and the generalized Lagrangian

$$L^g(x,\alpha,\lambda) := \alpha f(x) + \langle \lambda, G(x)\rangle, \quad (\alpha,\lambda) \in I\!R \times Y^*.$$

Let x_0 be a locally optimal solution of problem (P). Then, F. John type (first order) optimality conditions can be written in the form

$$D_x L^g(x_0,\alpha,\lambda) = 0,\ \alpha \geq 0,\ \lambda \in N_K(G(x_0)), \tag{3.1}$$

where α and λ are not both zero and $N_K(y) := \{y^* \in Y^* : \langle y^*, z - y\rangle \le 0,\ \forall\, z \in K\}$ is the normal cone to K at y. We denote by $\Lambda^g(x_0)$ the set of generalized Lagrange multipliers $(\alpha, \lambda) \ne (0, 0)$ satisfying condition (3.1). It should be noted that for a general Banach space Y the set $\Lambda^g(x_0)$ can be empty. The above F. John optimality condition is necessary for local optimality, i.e. $\Lambda^g(x_0) \ne \emptyset$, in two important cases, namely when the space Y is finite dimensional or when the set K has a non empty interior [15, 27].

If the multiplier α in (3.1) is non zero, then we can take $\alpha = 1$ and hence the corresponding first order necessary condition becomes

$$D_x L(x_0, \lambda) = 0,\ \lambda \in N_K(G(x_0)). \tag{3.2}$$

Under Robinson's constraint qualification (2.10) the set $\Lambda(x_0)$ of Lagrange multipliers satisfying (3.2) is nonempty and bounded [17, 27]. In case the set K is a convex *cone* and $y \in K$, the normal cone $N_K(y)$ can be written in the form $N_K(y) = \{y^* \in K^- : \langle y^*, y\rangle = 0\}$, where

$$K^- := \{y^* \in Y^* : \langle y^*, y\rangle \le 0,\ \forall\, y \in K\}$$

is the dual cone of the cone K. In this case condition $\lambda \in N_K(G(x_0))$ becomes $\lambda \in K^-$ and $\langle \lambda, G(x_0)\rangle = 0$.

Let us finally recall that the cone

$$C(x_0) := \{h \in X : DG(x_0)h \in T_K(G(x_0)), Df(x_0)h \le 0\} \tag{3.3}$$

is called the *critical cone* of the problem (P) at the point x_0. It represents those directions for which a first order linearization of (P) does not provide an information about optimality of x_0. It may be noted that when the set $\Lambda(x_0)$ of Lagrange multipliers is nonempty, then $Df(x_0)h \ge 0$ for any $h \in X$ satisfying $DG(x_0)h \in T_K(G(x_0))$. In such a case the inequality $Df(x_0)h \le 0$ in the definition of the critical cone can be replaced by the equation $Df(x_0)h = 0$, which in turn is equivalent to $\langle \lambda, DG(x_0)h\rangle = 0$ for any $\lambda \in \Lambda(x_0)$.

With these preliminaries we may now state our second order necessary condition for optimality, which is based on the analysis of parabolic paths of the form

$$x(t) = x_0 + th + \tfrac{1}{2}t^2 w + o(t^2). \tag{3.4}$$

This result improves [8, Theorem 4.2], where a similar theorem is stated based on the *inner* second order tangent set.

Theorem 3.1 *Let x_0 be a locally optimal solution of the problem (P). Suppose that Robinson's constraint qualification (2.10) holds. Then for all $h \in C(x_0)$,*

$$\sup_{\lambda \in \Lambda(x_0)} \left\{ D^2_{xx} L(x_0, \lambda)(h, h) - \sigma(\lambda, \mathcal{T}(h)) \right\} \ge 0, \tag{3.5}$$

where $\mathcal{T}(h) := \mathcal{T}^2_K(G(x_0), DG(x_0)h)$.

Proof. Note that if $\mathcal{T}(h) = \emptyset$, then $\sigma(\cdot, \mathcal{T}(h)) = -\infty$ and (3.5) trivially holds. Therefore we assume that the set $\mathcal{T}(h)$ is non empty.

We claim that the optimal value of the following optimization problem

$$\begin{array}{ll} \text{Min}_{w \in X} & Df(x_0)w + D^2 f(x_0)(h,h) \\ \text{subject to} & DG(x_0)w + D^2 G(x_0)(h,h) \in \mathcal{T}(h) \end{array} \tag{3.6}$$

is non-negative. Indeed if w is feasible for this problem, using Proposition 2.2 we get $w \in \mathcal{T}^2_{G^{-1}(K)}(x_0, h)$ so that we may find a sequence $t_k \downarrow 0$ such that $x_k := x_0 + t_k h + \frac{1}{2} t_k^2 w + o(t_k^2) \in G^{-1}(K)$. The sequence x_k is feasible for (P) and converges to the local minimum x_0 so that $f(x_k) \geq f(x_0)$ for all k sufficiently large. By using the second order Taylor expansion we have

$$f(x_0) \leq f(x_k) = f(x_0) + t_k Df(x_0)h + \tfrac{1}{2} t_k^2 [Df(x_0)w + D^2 f(x_0)(h,h)] + o(t_k^2),$$

and since $Df(x_0)h = 0$ for any $h \in C(x_0)$, we obtain

$$Df(x_0)w + D^2 f(x_0)(h,h) \geq 0$$

establishing our claim.

The optimization problem given in the left hand side of (3.5) is the (parametric) dual (cf. [20], [6]) of the problem (3.6). Indeed, the Lagrangian of (3.6) is

$$\mathcal{L}(w, \lambda) = D_x L(x_0, \lambda) w + D^2_{xx} L(x_0, \lambda)(h,h).$$

Also by (2.9) we have that $\sigma(\lambda, \mathcal{T}(h)) = +\infty$ for any $\lambda \notin [T_{T_K(y)}(d)]^-$, with $y = G(x_0)$ and $d = DG(x_0)h$. In particular $\sigma(\lambda, \mathcal{T}(h)) = +\infty$ for $\lambda \notin T_K(y)^- = N_K(y)$. It follows that the effective domain of the parametric dual of (3.6) is contained in $\Lambda(x_0)$. The duality then follows. Moreover, Robinson's constraint qualification (2.10) implies that

$$DG(x_0)X - T_K(G(x_0)) = Y.$$

Since for any $z \in \mathcal{T}(h)$, it follows that $z + T_K(G(x_0)) \subset \mathcal{T}(h)$, we have that

$$z + DG(x_0)X - \mathcal{T}(h) = Y.$$

Therefore (3.6) has a feasible solution and Robinson's constraint qualification for the problem (3.6) holds as well. Consequently there is no duality gap between (3.6) and its dual (cf. [6]), and hence (3.5) follows. ▮

Remarks. (i) The set $\mathcal{T}(h)$ in (3.5) and in (3.6) can be replaced by the topological closure of its convex hull. This will not change the corresponding support function. (ii) If K is second order directionally differentiable, that is $\mathcal{T}^2_K(G(x_0), DG(x_0)d) = T^2_K(G(x_0), DG(x_0)d)$, (3.5) is equivalent to [8, Theorem 4.2]. In general however, the set $\mathcal{T}^2_K(G(x_0), DG(x_0)d)$ is larger than $T^2_K(G(x_0), DG(x_0)d)$ and therefore Theorem 3.1 is stronger. (iii) If $0 \in \mathcal{T}(h)$ for every $h \in C(x_0)$, in particular if the set K is polyhedral, then $\mathcal{T}(h) = T_{T_K(G(x_0))}(DG(x_0)h)$ and $\sigma(\lambda, \mathcal{T}(h)) = 0$ for every $\lambda \in \Lambda(x_0)$. Therefore in that case the "sigma term" in (3.5) vanishes.

Definition 3.1 *Let $S \subset G^{-1}(K)$ be a set of feasible points of the problem (P) such that $f(x) = f_0$ for all $x \in S$. It is said that the second order growth condition holds at S if there exist a constant $c > 0$ and a neighborhood N of S such that*

$$f(x) \geq f_0 + c\, \mathrm{dist}(x, S)^2, \quad \forall\, x \in G^{-1}(K) \cap N. \tag{3.7}$$

In particular, if $S = \{x_0\}$ is a singleton, the second order growth condition (3.7) takes the form

$$f(x) \geq f(x_0) + c\|x - x_0\|^2, \quad \forall\, x \in G^{-1}(K) \cap N,$$

which clearly implies that x_0 is a locally optimal solution of (P). Moreover, in this case it follows that for any $h \in C(x_0)$ the optimal value of (3.6) is greater than or equal to $2c\|h\|^2$, so that the second order necessary condition (3.5) can be strengthen to strict inequality for all non zero $h \in C(x_0)$.

The above second order necessary condition is based on *upper* estimates of the objective function along parabolic paths of the form (3.4). In order to derive lower estimates, and hence to obtain second order *sufficient* conditions, we need an additional concept.

Definition 3.2 *Let $y \in K$, $d \in T_K(y)$, and consider a continuous linear mapping $M : X \to Y$. We say that a set $\mathcal{A}_K(y, d; M) \subset Y$ is an upper second order approximation set for K at the point y in the direction d and with respect to M, if for any sequence $y_k \in K$ of the form $y_k := y + t_k d + \frac{1}{2} t_k^2 r_k$, where $t_k \downarrow 0$ and $r_k = M w_k + a_k$ with $\{a_k\}$ being a convergent sequence in Y and $\{w_k\} \subset X$ satisfying $t_k w_k \to 0$, the following condition holds*

$$\lim_{k \to \infty} \mathrm{dist}\,(r_k, \mathcal{A}_K(y, d; M)) = 0. \tag{3.8}$$

If the above holds for any X and M, i.e. (3.8) is satisfied for any sequence $y + t_k d + \frac{1}{2} t_k^2 r_k \in K$ such that $t_k r_k \to 0$, we omit M and say that the set $\mathcal{A}_K(y, d)$ is an upper second order approximation set for K at the point y in the direction d.

Let us make the following observations. The upper second order approximation set $\mathcal{A}_K(y, d)$ is not unique. Clearly, if $\mathcal{A}_K(y, d) \subset B$ then B is also an upper second order approximation set. Since if $y \in K$, $d \in T_K(y)$ and $y + d + w \in K$ imply $d + w \in T_K(y)$ and hence $w \in T_{T_K(y)}(d)$, it follows that the set $T_{T_K(y)}(d)$ is always an upper second order approximation set. It is also not difficult to see from the definitions that the outer second order tangent set $\mathcal{T}_K^2(y, d)$ is included in any upper second-order approximation set $\mathcal{A}_K(y, d)$.

Theorem 3.2 *Let x_0 be a feasible point of the problem (P) satisfying the first order (F. John type) optimality condition (3.1). Let to every $h \in C(x_0)$ correspond an upper second order approximation set $\mathcal{A}_K(G(x_0), DG(x_0)h; DG(x_0))$, and suppose that the following second order condition is satisfied.*

$$\sup_{(\alpha,\lambda) \in \Lambda^g(x_0)} \left\{ D_{xx}^2 L^g(x_0, \alpha, \lambda)(h, h) - \sigma(\lambda, \mathcal{A}(h)) \right\} > 0, \; \forall\, h \in C(x_0) \setminus \{0\}, \tag{3.9}$$

where $\mathcal{A}(h) := \mathcal{A}_K(G(x_0), DG(x_0)h; DG(x_0))$. Then the second order growth condition holds at x_0, and hence x_0 is a strict locally optimal solution of (P).

Proof. We argue by contradiction. Suppose that the second order growth condition does not hold at x_0. Then there exists a sequence of feasible points $x_k \in G^{-1}(K)$ converging to x_0 and such that

$$f(x_k) \leq f(x_0) + o(t_k^2), \tag{3.10}$$

where $t_k := \|x_k - x_0\|$. Since the space X is finite dimensional, and hence bounded closed sets in X are compact, we can assume that $h_k := (x_k - x_0)/t_k$ converges to a vector $h \in X$. Clearly $\|h\| = 1$ and hence $h \neq 0$. By using first order Taylor expansions, we obtain from $G(x_k) \in K$ that $DG(x_0)h \in T_K(G(x_0))$ and from (3.10) that $Df(x_0)h \leq 0$. Therefore it follows that $h \in C(x_0)$.

By a second order Taylor expansion of $G(x_k)$ at x_0, we have that

$$G(x_k) = y_0 + t_k d + \tfrac{1}{2} t_k^2 \left(DG(x_0) w_k + D^2 G(x_0)(h,h) \right) + o(t_k^2),$$

where $y_0 := G(x_0)$, $d := DG(x_0)h$ and $w_k := 2t_k^{-2}(x_k - x_0 - t_k h)$. Note that $x_k - x_0 - t_k h = o(t_k)$ and hence $t_k w_k \to 0$. Together with the definition of upper second order approximation set this implies that

$$DG(x_0) w_k + D^2 G(x_0)(h,h) \in \mathcal{A}(h) + o(1) B_Y. \tag{3.11}$$

We also have that

$$f(x_k) = f(x_0) + t_k Df(x_0)h + \tfrac{1}{2} t_k^2 \left(Df(x_0) w_k + D^2 f(x_0)(h,h) \right) + o(t_k^2),$$

so that using (3.10) and (3.11), one can find a sequence $\varepsilon_k \to 0$ such that

$$\left\{ \begin{array}{l} 2t_k^{-1} Df(x_0)h + (Df(x_0) w_k + D^2 f(x_0)(h,h)) \leq \varepsilon_k, \\ DG(x_0) w_k + D^2 G(x_0)(h,h) \in \mathcal{A}(h) + \varepsilon_k B_Y. \end{array} \right. \tag{3.12}$$

By (3.9), there exists $(\alpha, \lambda) \in \Lambda^g(x_0)$ such that

$$D_{xx}^2 L^g(x_0, \alpha, \lambda)(h,h) - \sigma(\lambda, \mathcal{A}(h)) \geq \kappa \tag{3.13}$$

for some $\kappa > 0$. It follows from the second condition in (3.12) that

$$\langle \lambda, DG(x_0) w_k + D^2 G(x_0)(h,h) \rangle \leq \sigma(\lambda, \mathcal{A}(h) + \varepsilon_k B_Y) = \sigma(\lambda, \mathcal{A}(h)) + \varepsilon_k \|\lambda\|.$$

Also $\alpha \geq 0$, and if $\alpha \neq 0$, then there exists a Lagrange multiplier and hence $Df(x_0)h = 0$. In any case $\alpha Df(x_0)h = 0$, and hence we obtain from (3.12) and (3.13) that

$$\begin{array}{rcl} 0 & \geq & \alpha(2t_k^{-1} Df(x_0)h + Df(x_0) w_k + D^2 f(x_0)(h,h) - \varepsilon_k) + \\ & & \langle \lambda, DG(x_0) w_k + D^2 G(x_0)(h,h) \rangle - \sigma(\lambda, \mathcal{A}(h)) - \varepsilon_k \|\lambda\| \\ & = & D_{xx}^2 L^g(x_0, \alpha, \lambda)(h,h) - \sigma(\lambda, \mathcal{A}(h)) - \varepsilon_k(\alpha + \|\lambda\|) \\ & \geq & \kappa - \varepsilon_k(\alpha + \|\lambda\|). \end{array}$$

Since $\varepsilon_k \to 0$ we obtain a contradiction which completes the proof. ∎

Let us first observe that the *finite* dimensionality of the space X was used in the derivation of the above second order *sufficient* condition, while the corresponding second order necessary condition did not require that assumption.

If the set $\Lambda(x_0)$ of Lagrange multipliers is non empty, then the second order sufficient condition (3.9) is equivalent to

$$\sup_{\lambda\in\Lambda(x_0)} \left\{D^2_{xx}L(x_0,\lambda)(h,h) - \sigma(\lambda, \mathcal{A}(h))\right\} > 0, \quad \forall\, h \in C(x_0) \setminus \{0\}. \tag{3.14}$$

Also, as it was mentioned earlier, the set $\mathcal{Z}(h) := T_{T_K(G(x_0))}(DG(x_0)h)$ is always an upper second order approximation set. Furthermore,

$$\sigma(\lambda, \mathcal{Z}(h)) = \begin{cases} 0, & \text{if } \lambda \in T_K(G(x_0)) \text{ and } \langle \lambda, DG(x_0)h\rangle = 0, \\ +\infty, & \text{otherwise.}\end{cases}$$

Therefore for that choice of upper second order approximation set, the second order sufficient condition (3.9) takes the form

$$\sup_{(\alpha,\lambda)\in\Lambda^g(x_0)} D^2_{xx}L^g(x_0,\alpha,\lambda)(h,h) > 0, \quad \forall\, h \in C(x_0) \setminus \{0\}.$$

4 Second-order regularity

Comparing the necessary and sufficient conditions given in (3.5) and (3.14), one may observe that besides the change from weak to strict inequality, the set $\mathcal{T}(h) = \mathcal{T}^2_K(G(x_0), DG(x_0)h)$ in the former was replaced by a possibly larger set $\mathcal{A}(h)$. Now, condition (3.14) becomes stronger if one can take a smaller second order approximation set $\mathcal{A}(h)$. In particular, if $\mathcal{T}(h)$ is an upper second-order approximation set, condition (3.14) becomes the strongest possible by taking $\mathcal{A}(h) = \mathcal{T}(h)$, in which case the gap between (3.5) and (3.14) reduces to the difference between weak and strict inequality, just as in the case of mathematical programming. This motivates the following definition.

Definition 4.1 *We say that the set K is second order regular at a point $y \in K$ in a direction $d \in T_K(y)$ and with respect to a linear mapping $M : X \to Y$, if the outer second order tangent set $\mathcal{T}^2_K(y,d)$ is an upper second order approximation set for K at y in the direction d with respect to M. If K is second order regular at $y \in K$ in every direction $d \in T_K(y)$ and with respect to any X and M, we say that K is second order regular at y.*

Loosely speaking, second order regular sets are the appropriate ones for second order optimality conditions, in the sense that there is no gap between the corresponding second order necessary and sufficient conditions. Moreover, it follows from Theorems 3.1 and 3.2 that if Robinson's constraint qualification holds and the set K is second order regular at $G(x_0)$ in every direction $DG(x_0)h$ with $h \in C(x_0)$ and with respect to $M := DG(x_0)$, then the second order condition (3.14) with $\mathcal{A}(h) = \mathcal{T}(h)$ is *equivalent* to the second order growth condition at the point x_0.

At first glance the second order regularity concept, introduced in Definition 4.1, may seem to be rather technical. Nevertheless it is possible to verify the second order regularity in a number of particular situations. It holds, for example, when $0 \in \mathcal{T}_K^2(y,d)$ for every $d \in T_K(y)$, since then $\mathcal{T}_K^2(y,d) = T_{T_K(y)}(d)$. This occurs, for instance, when K is a polyhedral set. We discuss in the next subsections several other situations where the second order regularity holds. In particular, we show that the cone $\mathcal{S}_+^n$ of $n \times n$ positive semi-definite matrices is second order regular (at every point $y \in \mathcal{S}_+^n$).

4.1 Sets defined by smooth and convex constraints

Second order regularity is preserved when taking inverse images through twice continuously differentiable mappings satisfying Robinson's constraint qualification.

Proposition 4.1 *Let K be a closed convex subset of Y and $G : X \to Y$ be a twice continuously differentiable mapping. If Robinson's constraint qualification (2.10) holds and K is second order regular at $G(x_0)$ in the direction $DG(x_0)h$ with respect to the linear mapping $M := DG(x_0)$, then the set $G^{-1}(K)$ is second order regular at x_0 in the direction h.*

Proof. Let $x_k := x_0 + t_k h + \frac{1}{2}t_k^2 r_k \in G^{-1}(K)$ be such that $t_k \downarrow 0$ and $t_k r_k \to 0$. By Proposition 2.2 and Robinson-Ursescu stability theorem we obtain for some constant L and all k large enough,

$$\begin{aligned}
&\operatorname{dist}\left(r_k, \mathcal{T}^2_{G^{-1}(K)}(x_0,h)\right) \\
&\quad = \operatorname{dist}\left(r_k, DG(x_0)^{-1}\left[\mathcal{T}_K^2(G(x_0), DG(x_0)h) - D^2G(x_0)(h,h)\right]\right) \\
&\quad \leq L \operatorname{dist}\left(DG(x_0)r_k + D^2G(x_0)(h,h), \mathcal{T}_K^2(G(x_0), DG(x_0)h)\right).
\end{aligned}$$

Now, a second order expansion of $G(x_k)$ gives

$$G(x_k) = G(x_0) + t_k DG(x_0)h + \tfrac{1}{2}t_k^2\left(DG(x_0)r_k + D^2G(x_0)(h,h)\right) + o(t_k^2).$$

Since $G(x_k) \in K$, the assumed second order regularity of K implies

$$\operatorname{dist}\left(DG(x_0)r_k + D^2G(x_0)(h,h), \mathcal{T}_K^2(G(x_0), DG(x_0)h\right) \to 0$$

and therefore $\operatorname{dist}\left(r_k, \mathcal{T}^2_{G^{-1}(K)}(x_0,h)\right) \to 0$, as was to be proved. ∎

Since the second order regularity concept was defined for *convex* sets, strictly speaking, we should verify in Proposition 4.1 that the set $G^{-1}(K)$ is convex. Note, however, that convexity of $G^{-1}(K)$ was not used in the above proof in any way.

Consider the set

$$F := \{x \in X : g_i(x) \leq 0,\ i = 1,\dots,p;\ \ h_j(x) = 0,\ j = 1,\dots q\},$$

defined by a finite number of constraints. Suppose that the functions g_i and h_j are twice continuously differentiable. As a straightforward consequence of Proposition 4.1 and the fact that polyhedral sets are second order regular, we obtain that the set F is second order regular at every point $x_0 \in F$ satisfying the Mangasarian-Fromovitz constraint qualification. Another direct consequence of Proposition 4.1 is the following result.

Corollary 4.1 *Let $K_1, \dots, K_n$ be closed convex sets which are second order regular at a point $y_0 \in K_1 \cap \dots \cap K_n$ in a direction $d \in T_{K_1}(y_0) \cap \dots \cap T_{K_n}(y_0)$. If there exists a point in K_n which belongs to the interior of the remaining K_i's, $i = 1, \dots, n-1$, then the intersection $K_1 \cap \dots \cap K_n$ is second order regular at y_0 in the direction d.*

Proof. It suffices to apply Proposition 4.1 with $G : Y \to Y^n$ given by $G(y) = (y, \dots, y)$, and $K = K_1 \times \dots \times K_n$. It is easily seen that K is second order regular at $(y_0, \dots, y_0)$ in the direction $(d, \dots, d)$.

In order to check Robinson's constraint qualification we take $\bar{y} \in Y$ and $\varepsilon > 0$ such that $\bar{y} \in K_n$ and $\bar{y} + 2\varepsilon B_Y \subset K_1 \cap \dots \cap K_{n-1}$. If $u_1, \dots, u_n \in \varepsilon B_Y$, letting $y = \bar{y} + u_n$ we have $k_i := y - u_i \in \bar{y} + 2\varepsilon B_Y \subset K_i$ for all $i = 1, \dots, n-1$. Therefore, if we set $k_n := \bar{y} \in K_n$ we have $u_i = y - k_i \in y - K_i$ for all $i = 1, \dots, n$ and then $[\varepsilon B_Y]^n \subset G(Y) - K$ which proves Robinson's constraint qualification. ∎

Returning to the case of sets defined by inequality constraints, we observe that when the constraint functions are convex one may relax the differentiability assumptions.

Proposition 4.2 *Let $K := \{y : h(y) \le 0\}$, where $h(\cdot)$ is a convex function which is continuous in a neighborhood of y_0. Suppose that the Slater condition holds and that $h(y_0) = 0$. Then K is second-order regular at y_0 if and only if, for any $d \in T_K(y_0)$ satisfying $h'(y_0, d) = 0$, and any path $y(t) \in K$ of the form $y(t) = y_0 + td + \frac{1}{2}t^2 r(t)$, $t \ge 0$, with $tr(t) \to 0$ as $t \downarrow 0$, the inequality*

$$\limsup_{t \downarrow 0} h''_-(y_0; d, r(t)) \le 0 \tag{4.1}$$

holds.

Proof. Consider a direction $d \in T_K(y_0)$ and a sequence $y_k := y_0 + t_k d + \frac{1}{2}t_k^2 r_k \in K$, with $t_k \downarrow 0$ and $t_k r_k \to 0$. It follows from $d \in T_K(y_0)$ that $h'(y_0, d) \le 0$. Since $h'(y_0, d) < 0$ implies $\mathcal{T}_K^2(y_0, d) = Y$, we only need to consider the case $h'(y_0, d) = 0$.

Because of the Slater condition, there is a point $\bar{y} \in Y$ such that $h(\bar{y}) < 0$. By convexity of $h(\cdot)$ we then have that $h(y_0 + t(\bar{y} - y_0)) < 0$ for any $t \in (0, 1)$, and hence a point $\bar{y}$ where $h(\bar{y}) < 0$ can be chosen arbitrary close to y_0. Therefore we can assume that $h(\cdot)$ is continuous at $\bar{y}$.

Assume that (4.1) holds. Proceeding as in the proof of Proposition 2.1 ((2.7) to be precise), it follows that for $\alpha > 0$ and $w_\alpha := r_k + \alpha(\bar{y} - y_0)$ one has

$$h''_-(y_0; d, w_\alpha) \le h''_-(y_0; d, r_k) + \alpha h(\bar{y}).$$

and by virtue of (4.1) we deduce $h''_-(y_0; d, r_k + \alpha(\bar{y} - y_0)) < 0$ for all k sufficiently large. Proposition 2.1 implies that $r_k + \alpha(\bar{y} - y_0) \in \mathcal{T}^2_K(y_0, d)$ so that

$$\limsup_{k\to\infty} \ \operatorname{dist}(r_k, \mathcal{T}^2_K(y_0, d)) \le \alpha\|\bar{y} - y_0\|.$$

Since α can be made arbitrarily small, we obtain that K is second-order regular.

Conversely, assume that K is second-order regular. Let $t_k \downarrow 0$ be a sequence through which the upper limit (4.1) is attained as a limit, and let $r_k := r(t_k)$. Set $\varepsilon_k := \operatorname{dist}(r_k, \mathcal{T}^2_K(y_0, d)) + 1/k$, so that $\varepsilon_k \to 0$, and choose $\tilde{r}_k \in \mathcal{T}^2_K(y_0, d)$ such that $\|r_k - \tilde{r}_k\| < \varepsilon_k$. Then, for some sequence $\tau_\ell \downarrow 0$ we have $y_0 + \tau_\ell d + \frac{1}{2}\tau_\ell^2 \tilde{r}_k + o(\tau_l^2) \in K$ and therefore $y_0 + \tau_\ell d + \frac{1}{2}\tau_\ell^2 r_k \in K + \frac{1}{2}\varepsilon_k \tau_\ell^2 B_Y$. Then, for all $\alpha > 0$ and $w_\alpha := r_k + \alpha(\bar{y} - y_0)$ we get

$$\begin{aligned} y_0 + \tau_\ell d + \tfrac{1}{2}\tau_\ell^2 w_\alpha &= (1 - \tfrac{1}{2}\alpha\tau_\ell^2)(y_0 + \tau_\ell d + \tfrac{1}{2}\tau_\ell^2 r_k) + \tfrac{1}{2}\alpha\tau_\ell^2(\bar{y} + \tau_\ell d + \tfrac{1}{2}\tau_\ell^2 r_k) \\ &\subset (1 - \tfrac{1}{2}\alpha\tau_\ell^2)(K + \tfrac{1}{2}\varepsilon_k\tau_\ell^2 B_Y) + \tfrac{1}{2}\alpha\tau_\ell^2(\bar{y} + \tau_\ell d + \tfrac{1}{2}\tau_\ell^2 r_k) \\ &= (1 - \tfrac{1}{2}\alpha\tau_\ell^2)K + \tfrac{1}{2}\alpha\tau_\ell^2\left[\bar{y} + \tau_\ell d + \tfrac{1}{2}\tau_\ell^2 r_k + (1 - \tfrac{1}{2}\alpha\tau_\ell^2)\frac{\varepsilon_k}{\alpha} B_Y\right]. \end{aligned}$$

Since $\bar{y} \in \operatorname{int}K$, for k large enough, $\bar{y} + 2\varepsilon_k\alpha^{-1}B_Y \subset K$, whence $y_0 + \tau_\ell d + \frac{1}{2}\tau_\ell^2 w_\alpha \in K$. By Proposition 2.1, $h''_-(y_0, d, w_\alpha) \le 0$. Since h is continuous at y_0, it is locally Lipschitz continuous. Therefore, $h''_-(y_0, d, \cdot)$ is globally Lipschitz continuous with the same constant, say L, and $h''_-(y_0, d, r_k) \le L\|w_\alpha - r_k\| = \alpha L\|\bar{y} - y_0\|$, whence

$$\limsup_{t\downarrow 0} h''_-(y_0, d, r(t)) = \lim_k h''_-(y_0, d, r_k) \le \alpha L\|\bar{y} - y_0\|.$$

Since α may be taken arbitrarily small, the conclusion follows. ∎

Let us derive now some criteria which allow to check condition (4.1). We first observe that this condition is satisfied whenever

$$h(y_0 + td + \tfrac{1}{2}t^2 r(t)) \ge h(y_0) + th'(y_0, d) + \tfrac{1}{2}t^2 h''_-(y_0; d, r(t)) + o(t^2), \tag{4.2}$$

for all $r(t)$ such that $tr(t) \to 0$ as $t \downarrow 0$. This holds for instance when h is twice continuously differentiable.

Note also that if the functions $\{h_i : i = 1, ..., m\}$ are second order directionally differentiable and satisfy (4.1), then $h := \sum_{i=1}^n h_i$ satisfies (4.1) as well. The same holds for $h(y) := \max\{h_i(y) : 1 \le i \le m\}$ which (combined with Slater condition) implies second order regularity of the set

$$K = \{y \in Y : h_i(y) \le 0, \ i = 1, ..., m\}.$$

To prove the latter, consider a point y_0 such that $h(y_0) = 0$. It is not difficult to show that the max-function is second order directionally differentiable with

$$\begin{aligned} h'(y_0, d) &= \max\{h_i'(y_0, d) : i \in I_1(y_0)\}, \\ h''(y_0; d, w) &= \max\{h_i''(y_0; d, w) : i \in I_2(y_0, d)\}, \end{aligned}$$

where $I_1(y) := \{i : h_i(y) = h(y)\}$ and $I_2(y) := \{i \in I_1(y) : h_i'(y,d) = h'(y,d)\}$. Since $h_i(\cdot)$ satisfy (4.1), we have that for $y(t) := y_0 + td + \frac{1}{2}t^2 r(t)$, such that $h(y(t)) \le 0$, $tr(t) \to 0$, and for d satisfying $h'(y_0,d) = 0$,

$$h''(y_0;d,r(t)) = \max_{i \in I_2(y_0,d)} h_i''(y_0;d,r(t)) \le o(t^2).$$

It then follows, assuming the Slater condition to hold (i.e., there exists $\bar{y}$ such that $h_i(\bar{y}) < 0$, $i = 1,...,m$), that the set K is second order regular and second order directionally differentiable with

$$T_K^2(y_0,d) = \mathcal{T}_K^2(y_0,d) = \{w \in Y : h_i''(y_0,d,w) \le 0,\ i \in I_2(y_0,d)\}.$$

A nondifferentiable (at zero) function satisfying (4.2) is the Euclidean norm $h(y) := \|y\|$. Many problems of robust optimization boil down to the minimization of a sum of Euclidean norms subject to linear constraints (see e.g. [2]), say $\sum_{i=1}^m \|A_i^t x\|$, where A_i are $q_i \times n$ matrices. Let us consider for simplicity the unconstrained problem. Introducing slack variables z_i, the problem reduces to the minimization of $\sum_{i=1}^m z_i$, subject to the constraints

$$\|A_i^t x\| - z_i \le 0, \quad 1 \le i \le m.$$

Set $h(y) = \|y\|$. Note that this is a twice continuously differentiable function at $y_0 \ne 0$. If $y_0 = 0$, we obtain $h'(0,d) = \|d\|$. If $d = 0$ then $h''(0;,d,w) = \|w\|$, otherwise $h''(0;d,w) = \langle d,w\rangle/\|d\|$. In both cases (4.2) is easily checked. Therefore $h_i(x,z) = \|A_i^t x\| - z_i$ satisfies also (4.2), and Slater condition is trivially satisfied.

As we mentioned in the introduction, an alternative approach to derivation of second order optimality conditions is to consider composite functions as in the problem (1.2), and that such problems can be investigated in the form (1.3). In this transformation the corresponding convex function is replaced by its epigraph. This motivates the following definition.

Definition 4.2 *Let $g(y)$ be a convex function with a finite value at a point $y_0 \in Y$. We say that $g(\cdot)$ is second order regular at y_0 if the set $K := \operatorname{epi}(g)$ is second order regular at the point $(y_0, g(y_0))$.*

The set $\operatorname{epi}(g)$ is defined by the constraint $h(y,c) \le 0$ where $h(y,c) := g(y) - c$, for which Slater condition holds. Therefore Propositions 2.1 and 4.2 imply the following result.

Proposition 4.3 *Let $g : Y \to \mathbb{R} \cup \{\infty\}$ be a closed proper convex function. If g is finite and continuous at a point $y_0 \in Y$, then g is second order regular at y_0 if and only if, for every $d \in Y$ and every path $r : \mathbb{R}_+ \to Y$ satisfying $tr(t) \to 0$ as $t \downarrow 0$, we have*

$$g(y_0 + td + \tfrac{1}{2}t^2 r(t)) \ge g(y_0) + tg'(y_0,d) + \tfrac{1}{2}t^2 g_-''(y_0;d,r(t)) + o(t^2) \qquad (4.3)$$

holds.

4.2 Semi-infinite and semi-definite programming

Let us finally consider the case of semi-infinite programming with $Y := C(\Omega)$ and $K := C_+(\Omega)$, with Ω a compact metric space. For a function $y \in C_+(\Omega)$ its *contact set* is defined as

$$\Delta(y) := \{\omega \in \Omega : y(\omega) = 0\}, \tag{4.4}$$

and it is well known that $d \in T_K(y)$ iff $d(\omega) \geq 0$ for all $\omega \in \Delta(y)$ (e.g., [23]). Denote

$$\Delta^*(y, d) := \{\omega \in \Delta(y) : d(\omega) = 0\}. \tag{4.5}$$

Note that if the set $\Delta^*(y, d)$ is empty, then $y + td \in \operatorname{int}\{C_+(\Omega)\}$ for all $t > 0$ small enough, and hence in that case $T_K^2(y, d) = Y$.

Suppose that Ω is a smooth compact manifold in a finite dimensional vector space. Consider a twice continuously differentiable function $y \in K$ with a non empty contact set, and a function $d \in T_K(y)$. Suppose further that $\Delta(y)$ is a smooth submanifold of Ω. A general formula for $T_K^2(y, d)$ is given in [9]. We derive now a particular case of that formula by direct arguments (which are considerably simpler) and, moreover, show that in the present case the second order regularity holds. These derivations are similar to the analyses in [22] and [6, part III].

By $T_1(\omega)$ we denote the tangent space to Ω at $\omega \in \Omega$ and by $T_2(\omega) \subset T_1(\omega)$ the tangent space to $\Delta(y)$ restricted to $T_1(\omega)$. Let $N(\omega)$ be the normal complement of $T_2(\omega)$ in $T_1(\omega)$, i.e. $N(\omega)$ is a linear space orthogonal to $T_2(\omega)$ and such that $T_2(\omega) + N(\omega) = T_1(\omega)$. Let $V(\omega)$ be a matrix whose columns form a basis of the linear space $N(\omega)$. Let us remark that the following second order growth condition

$$y(\omega) \geq c \operatorname{dist}(\omega, \Delta(y))^2, \quad \forall\, \omega \in \Omega \cap \mathcal{N}, \tag{4.6}$$

where $c > 0$ and $\mathcal{N}$ is a neighborhood of $\Delta(y)$, holds iff the matrix

$$U(\omega) := V(\omega)^t \nabla^2 y(\omega) V(\omega) \tag{4.7}$$

is positive definite for every $\omega \in \Delta(y)$ (see [22]).

Theorem 4.1 *Let $y \in K := C_+(\Omega)$ be a twice continuously differentiable function and let $d \in T_K(y)$ be continuously differentiable. Suppose that the set Ω is a smooth compact manifold, that $\Delta(y)$ is a smooth submanifold of Ω and that the second order growth condition (4.6) holds. Then the set K is second order directionally differentiable at y in the direction d, with*

$$T_K^2(y, d) = \left\{h \in C(\Omega) : h(\omega) \geq A(\omega)^t [U(\omega)]^{-1} A(\omega), \ \ \forall\, \omega \in \Delta^*(y, d)\right\}. \tag{4.8}$$

where $A(\omega) := V(\omega)^t \nabla d(\omega)$ and $U(\omega)$ is given in (4.7).

Moreover, let $M(x) := \sum_{i=1}^m x_i \psi_i(\cdot)$ be a linear mapping from $\mathbb{R}^m$ into $C(\Omega)$ such that the functions $\psi_i(\cdot)$, $i = 1, ..., m$, are Lipschitz continuous on Ω. Then the set K is second order regular at y in the direction d and with respect to M.

Proof. Since Ω is a smooth manifold, locally Ω can be identified with an open subset of a finite dimensional space, say $I\!R^p$. Therefore we can assume without loss of generality that $\Omega = I\!R^p$. Moreover, as we already observed, when $\Delta^*(y,d)$ is empty we have $\mathcal{T}^2_K(y,d) = T^2_K(y,d) = Y$ and the result holds trivially, so we may also assume that $\Delta^*(y,d) \neq \emptyset$.

Consider a path $\bar{y}_t(\cdot) := y(\cdot) + td(\cdot) + \frac{1}{2}t^2 h(\cdot)$ and the corresponding min-function $\nu(t) := \min_{\omega\in\Omega} \bar{y}_t(\omega)$. Since $\operatorname{dist}(\bar{y}_t, K) = \max\{0, -\nu(t)\}$, we have $h \in T^2(y,d)$ iff $\liminf_{t\downarrow 0} \nu(t)/t^2 \geq 0$, and $h \in \mathcal{T}^2(y,d)$ iff $\limsup_{t\downarrow 0} \nu(t)/t^2 \geq 0$. We shall prove that in fact the limit $\lim_{t\downarrow 0} \nu(t)/t^2$ exists so that both second order tangent sets coincide.

Let $\bar{\omega}_t$ be a minimizer of $\bar{y}_t(\omega)$ over Ω and let $\hat{\omega}_t$ be a point in $\Delta(y)$ closest (in the Euclidean norm) to $\bar{\omega}_t$. Note that $\delta_t := t^{-1}(\bar{\omega}_t - \hat{\omega}_t)$ is orthogonal to $\Delta(y)$ at the point $\hat{\omega}_t$, i.e. $\delta_t \in N(\hat{\omega}_t)$. Since $\Delta^*(y,d) \neq \emptyset$ we have $\nu(t) \leq O(t^2)$ and because of the second order growth condition (4.6) we get

$$\|\bar{\omega}_t - \hat{\omega}_t\| = \operatorname{dist}(\bar{\omega}_t, \Delta(y)) = O(t), \tag{4.9}$$

and also that $\operatorname{dist}(\bar{\omega}_t, \Delta^*(y,d)) \to 0$ as $t \to 0$ (see [22]).

By expanding $\bar{y}_t(\bar{\omega}_t)$ at $\hat{\omega}_t$, and since $y(\hat{\omega}_t) = 0$ and $\nabla y(\hat{\omega}_t) = 0$, we obtain

$$\nu(t) = \bar{y}_t(\bar{\omega}_t) = \bar{y}_t(\hat{\omega}_t) + \tfrac{1}{2}t^2\nabla^2 y(\hat{\omega}_t)(\delta_t, \delta_t) + t^2 \nabla d(\hat{\omega}_t)\delta_t + o(t^2).$$

Since $y(\hat{\omega}_t) = 0$ and $d(\hat{\omega}_t) \geq 0$, it follows that

$$\nu(t) \geq \tfrac{1}{2}t^2 h(\hat{\omega}_t) + \tfrac{1}{2}t^2\nabla^2 y(\hat{\omega}_t)(\delta_t, \delta_t) + t^2 \nabla d(\hat{\omega}_t)\delta_t + o(t^2). \tag{4.10}$$

Consequently, since $\operatorname{dist}(\hat{\omega}_t, \Delta^*(y,d)) \to 0$, we get

$$\liminf_{t\downarrow 0} \frac{\nu(t)}{t^2/2} \geq \min_{\omega\in\Delta^*(y,d)} \min_{\delta\in N(\omega)} \{h(\omega) + \nabla^2 y(\omega)(\delta,\delta) + 2\nabla d(\omega)\delta\}. \tag{4.11}$$

On the other hand, for any $\omega \in \Delta^*(y,d)$ and $\delta \in N(\omega)$ we have that $\nu(t) \leq \bar{y}_t(\omega + t\delta)$. By expanding the right hand side of this inequality, and since $y(\omega) = 0$, $\nabla y(\omega) = 0$, $d(\omega) = 0$ and $h(\omega + t\delta) = h(\omega) + o(1)$, we obtain that

$$\nu(t) \leq \tfrac{1}{2}t^2\{h(\omega) + \nabla^2 y(\omega)(\delta,\delta) + 2\nabla d(\omega)\delta\} + o(t^2), \tag{4.12}$$

which combined with (4.11) leads to

$$\lim_{t\downarrow 0} \frac{\nu(t)}{t^2/2} = \min_{\omega\in\Delta^*(y,d)} \min_{\delta\in N(\omega)} \{h(\omega) + \nabla^2 y(\omega)(\delta,\delta) + 2\nabla d(\omega)\delta\}. \tag{4.13}$$

It follows that $\mathcal{T}^2_K(y,d) = T^2_K(y,d)$ and $h \in T^2_K(y,d)$ iff for every $\omega \in \Delta^*(y,d)$,

$$h(\omega) + \min_{\delta\in N(\omega)} \{\nabla^2 y(\omega)(\delta,\delta) + 2\nabla d(\omega)\delta\} \geq 0. \tag{4.14}$$

By calculating the minimum in (4.14) we obtain (4.8). We point out that, because of the second order growth condition (4.6), the minimum on $\delta \in N(\omega)$ is attained for $\|\delta\| \leq \|\nabla d\|_\infty / c$.

The proof of second order regularity involves similar arguments. Let $\Psi(\omega) := (\psi_1(\omega), ..., \psi_m(\omega))^t$, and consider $t_k \downarrow 0$ and $y_k(\cdot) := y(\cdot) + t_k d(\cdot) + \frac{1}{2} t_k^2 h_k(\cdot) \in K$, where $h_k \in C(\Omega)$ are such that $h_k(\cdot) = x_k^t \Psi(\cdot) + a_k(\cdot)$ with $C(\Omega) \ni a_k \to a$ and $t_k x_k \to 0$. Consider also $\nu_k := \min_{\omega \in \Omega} y_k(\omega)$. Similar to (4.12) we have that for every $\omega \in \Delta^*(y, d)$ and $\delta \in N(\omega)$,

$$\nu_k \leq \tfrac{1}{2} t_k^2 \left[h_k(\omega + t_k \delta) + \nabla^2 y(\omega)(\delta, \delta) + 2\nabla d(\omega)\delta \right] + o(t_k^2). \tag{4.15}$$

Moreover, since $t_k x_k \to 0$ and $\Psi(\cdot)$ is Lipschitz continuous on Ω we have that

$$t_k x_k^t \Psi(\omega + t_k \delta) = t_k x_k^t \Psi(\omega) + o(t_k).$$

Also $a_k(\omega + t_k \delta) = a_k(\omega) + o(1)$ and hence

$$t_k^2 h_k(\omega + t_k \delta) = t_k^2 h_k(\omega) + o(t_k^2). \tag{4.16}$$

Since $y_k \in K$ and hence $\nu_k \geq 0$ we obtain then from (4.15) and (4.16) that

$$h_k(\omega) + \nabla^2 y(\omega)(\delta, \delta) + 2\nabla d(\omega)\delta + o(1) \geq 0, \tag{4.17}$$

where the term $o(1)$ can be taken uniformly in $\omega \in \Delta^*(y, d)$ and $\|\delta\| \leq \|\nabla d\|_\infty / c$. By using formula (4.14), we obtain from (4.17) that $h_k + o(1) \in \mathcal{T}_K^2(y, d)$, which completes the proof. ∎

It follows from the above theorem that for semi-infinite programs with constraints of the form $g(x, \omega) \geq 0$, $\omega \in \Omega$, there is no gap between the corresponding second order necessary and sufficient conditions under the following conditions:

(i) $g(\cdot, \omega)$ is twice differentiable with $\nabla^2_{xx} g(x, \omega)$ being continuous on $X \times \Omega$,

(ii) Robinson's constraint qualification holds,

(iii) $g(x_0, \cdot)$ satisfies the second order growth condition (4.6),

(iv) Ω is a smooth compact manifold and $\Delta(g(x_0, \cdot))$ a smooth submanifold of Ω,

(v) $g(x_0, \cdot)$ is twice continuously differentiable and the functions $\psi_i(\cdot) = \frac{\partial g}{\partial x_i}(x_0, \cdot)$ are continuously differentiable.

Note that since Ω is compact, the last assumption (v) implies that the functions $\psi_i(\cdot)$ are Lipschitz continuous on Ω. Also in the case of semi-infinite programming, Robinson's constraint qualification (postulated in the above condition (ii)) is equivalent to the extended Mangasarian-Fromovitz condition, that is, there exists $h \in X$ such that $h^t \nabla_x g(x_0, \omega) > 0$ for all $\omega \in \Delta_0 := \Delta(g(x_0, \cdot))$ (e.g., [23]). We also observe that when the function $g(\cdot, \omega)$ is concave for every fixed $\omega \in \Omega$, the feasible set

$$\Phi := \{x \in X : g(x, \omega) \geq 0,\ \forall\, \omega \in \Omega\}$$

is convex and Robinson's constraint qualification is equivalent to Slater condition: there exists $\bar{x} \in X$ such that $g(\bar{x}, \omega) > 0$ for all $\omega \in \Omega$.

Combining Theorem 4.1 with Propositions 2.2 and 4.1 we deduce that, under assumptions (i)–(v) above, the set Φ is second order regular at x_0 and also second order directionally differentiable with

$$T^2_\Phi(x_0, h) = \left\{ \begin{array}{l} u \in X : \nabla_x g(x_0, \omega) u + \\ \min_{\delta \in N(\omega)} \nabla^2 g(x_0, \omega)((h, \delta), (h, \delta)) \geq 0, \ \forall \omega \in \Delta_1(h) \end{array} \right\}, \tag{4.18}$$

where $\Delta_0 := \Delta(g(x_0, \cdot))$ and $\Delta_1(h) := \{\omega \in \Delta_0 : \nabla_x g(x_0, \omega) h = 0\}$. This formula may also be derived from Proposition 2.1, by using the characterization of second order directional derivatives of the min-function $\varphi(x) := \min_{\omega \in \Omega} g(x, \omega)$ given in [22, Theorem 4.1].

As an application, consider the example of semi-definite programming where

$$K = \mathcal{S}^n_+ := \{Z \in \mathcal{S}^n : g(Z, \omega) \geq 0, \ \forall \omega \in \Omega\},$$

with $g(Z, \omega) := \omega^t Z \omega$ and $\Omega := \{\omega \in I\!R^n : \|\omega\| = 1\}$. In this example the set Ω is a sphere, hence a smooth compact manifold. For a positive semi-definite matrix Z the corresponding contact set $\Delta(Z) := \{\omega \in \Omega : \omega^t Z \omega = 0\}$ is given by $\{\omega \in \Omega : Z\omega = 0\}$ which is a smooth submanifold of Ω. It is also not difficult to show that the corresponding second order growth condition holds (cf. [22]) and that the Lipschitz condition on functions ψ_i is automatically satisfied. Therefore we obtain the following result.

Corollary 4.2 *The cone $\mathcal{S}^n_+$ of symmetric positive semi-definite $n \times n$ matrices is second order regular and second order directionally differentiable at every point $Z \in \mathcal{S}^n_+$.*

An expression for the second order tangent sets and the corresponding second order optimality conditions for semi-definite optimization problems, are given explicitly in [24].

5 Extensions to nonisolated minima

Little is known about second order optimality conditions for nonisolated minima. A characterization of the second order growth condition is given in [5] for smooth qualified convex optimization problems with finitely many constraints. In [4] some sufficients conditions are stated for nonlinear programming problems. It is relatively easy to formulate a second-order necessary condition that generalizes a result in [4].

Let $S \subset G^{-1}(K)$ be a set of optimal solutions (minimizers) of the problem (P), and let $\mathcal{T}_S(x) := \limsup_{t \downarrow 0} t^{-1}(S - x)$ be the contingent cone to S at x. Clearly if $x \in S$ and $h \notin \mathcal{T}_S(x)$, then $\operatorname{dist}(x + th, S) \geq \beta t$ for some $\beta > 0$ and $t > 0$ small enough. Suppose that Robinson's constraint qualification holds at every point $x \in S$ and that S is compact. We have that if the second order growth condition holds at S, then for any feasible path $x(t)$ of the form (3.4), with $x_0 \in S$ and $h \notin \mathcal{T}_S(x_0)$, $f(x(t)) \geq f(x_0) + \gamma t^2$ for some $\gamma > 0$ and $t > 0$ small enough. It then follows by the arguments used in the proof of Theorem

3.1 that a necessary condition for the second order growth (at S) is that, for all $x \in S$ and $h \in C(x) \setminus \mathcal{T}_S(x)$,

$$\sup_{\lambda \in \Lambda(x)} \{D^2_{xx}L(x,\lambda)(h,h) - \sigma(\lambda, \mathcal{T}(x,h))\} > 0, \tag{5.1}$$

where $\mathcal{T}(x,h) := \mathcal{T}^2_K(G(x), DG(x)h)$.

Definition 5.1 *We say that the set S satisfies a property of uniform approximation of critical cones if for every $\varepsilon > 0$ there exists $\alpha > 0$ such that for all $x \in S$ and $h \in X$ satisfying $Df(x)h \leq \alpha\|h\|$ and $DG(x)h \in T_K(G(x)) + \alpha\|h\|B_Y$, we have $\operatorname{dist}(h, C(x)) \leq \varepsilon\|h\|$.*

Definition 5.2 *We say that K is uniformly regular with respect to the set S and the mapping $G(x)$ if for $x \in S$ and $h \in C(x)$, $\mathcal{T}^2_K(G(x), DG(x)h)$ is an upper second order approximation set for K at the point $G(x)$ in the direction $DG(x)h$ with respect to $DG(x)$, uniformly over S. That is, if $x_k \in S$, $h_k \in C(x_k)$, $t_k \downarrow 0$, and $r_k = DG(x_k)z_k + a_k$ are sequences such that $\{a_k\}$ is convergent, $t_k z_k \to 0$ and $G(x_k) + t_k DG(x_k)h_k + \frac{1}{2}t_k^2 r_k \in K$, then*

$$\lim_{k\to\infty} \operatorname{dist}(r_k, \mathcal{T}^2_K(G(x_k), DG(x_k)h_k)) = 0. \tag{5.2}$$

Theorem 5.1 *Let $S \subset G^{-1}(K)$ satisfy the property of uniform approximation of critical cones and suppose that Robinson's constraint qualification holds at every point $x \in S$, that S is compact and that K is uniformly regular with respect to the set S and the mapping $G(x)$. Then condition (5.1) is necessary and sufficient for the second order growth at S.*

Proof. We already observed that the condition is necessary. It suffices therefore to prove that it is sufficient. Let x_k be a sequence of feasible points $x_k \in G^{-1}(K)$ converging to a point $x_0 \in S$ and such that (3.10) holds. Let $\hat{x}_k$ be a projection of x_k onto S, i.e. $\hat{x}_k \in S$ and $\|x_k - \hat{x}_k\| = \operatorname{dist}(x_k, S)$. Set $t_k := \|x_k - \hat{x}_k\|$ and $\hat{h}_k := (x_k - \hat{x}_k)/t_k$. From the property of uniform approximation of critical cones, there exists h_k such that $h_k \in C(\hat{x}_k)$ and $\|h_k - \hat{h}_k\| \to 0$. Then $x_k = \hat{x}_k + t_k h_k + o(t_k)$. The remainder of the proof is similar to the one of Theorem 3.2. ∎

It was proved in [5] that the property of uniform approximation of critical cones is satisfied for finitely constrained convex optimization problems. Whether this property holds in more general settings is an open problem.

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